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Unboundedness of Solutions of Parabolic Differential Inequalities

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1. INTRODUCTION

On the basis of Sturmian comparison theory of elliptic equations, McNabb [6] developed criteria for unboundedness of solutions of second-order parabolic differential equations. His results were later extended by Dunninger [4] and Chan and Young [1] to second-order scalar and matrix differential inequalities of parabolic type. More recently, Chan and Young [2] have given related results for a class of fourth-order matrix differential inequalities.

In this paper we shall proceed further in this direction to obtain criteria for unboundedness of solutions of higher-order scalar and matrix parabolic differential inequalities in one space variable. The basis of our results is the Picone-type identities due to Kusano and Yoshida [5] for self-adjoint ordinary differential operators of arbitrary even order.

2. THE CASE OF SCALAR DIFFERENTIAL INEQUALITIES

Let (x_1, x_2) be a bounded interval on the real line. We denote by Q the semi-infinite strip $\{(x, t): x \in (x_1, x_2), t > 0\}$ and by $\bar{Q}$ its closure: $\{(x, t): x \in [x_1, x_2], t \geqslant 0\}$. Differentiation with respect to x (or t) is denoted by D_x (or D_t).

We consider the parabolic partial differential operator

$$lu \equiv \sum_{j=0}^{n} (-1)^j D_x^j(a_j(x, t) D_x^j u) + D_t u,$$

and the ordinary differential operator

$$mv \equiv \sum_{j=0}^{n} (-1)^j D_x^j(b_j(x) D_x^j v),$$

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where the coefficients a_j, b_j are real functions such that $a_j \in C^j(\bar{Q})$, $b_j \in C^j[x_1, x_2]$, $j = 0, 1, \ldots, n$, and $a_n > 0$, $b_n > 0$. The domain $\mathfrak{D}_\ell$ of ℓ is the set of all real functions u that are continuous in $\bar{Q}$ together with their first $2n$ x-derivatives and first t-derivatives, while the domain $\mathfrak{D}_m$ of m is the set of all real functions v of class $C^{2n}[x_1, x_2]$.

THEOREM 1. *Let $u \in \mathfrak{D}_\ell$ satisfy the following conditions:*

(i) $u\ell u \geq 0$ *in* Q,

(ii) $(-1)^k D_x{}^k u \sum_{j=k}^{n} (-1)^j D_x^{j-k}(a_j D_x{}^j u) \geq 0$ *in* Q, $\quad k = 1, \ldots, n-1$,

(iii) *none of* $u, D_x u, \ldots, D_x^{n-1} u$ *vanish in* $\bar{Q}$. (1)

If there exists a function $v \in \mathfrak{D}_m$ which satisfies

$$\text{(i)} \quad \int_{x_1}^{x_2} v m v \, dx \leq 0,$$

$$\text{(ii)} \quad v = D_x v = \cdots = D_x^{n-1} v = 0 \quad \text{at} \quad x_1 \text{ and } x_2, \tag{2}$$

$$\text{(iii)} \quad \int_0^\tau \int_{x_1}^{x_2} \sum_{j=0}^{n} (a_j - b_j)(D_x{}^j v)^2 \, dx \, dt \to -\infty \quad \text{as} \quad \tau \to \infty,$$

then u is unbounded in Q.

Proof. In view of (iii) of (1) we are able to apply a generalized Picone identity given in [5, Theorem 1A] to obtain

$$D_x \sum_{k=0}^{n-1} (-1)^k D_x{}^k v \left[\frac{D_x{}^k v}{D_x{}^k u} \sum_{j=k+1}^{n} (-1)^j D_x^{j-k-1}(a_j D_x{}^j u) - \sum_{j=k+1}^{n} (-1)^j D_x^{j-k-1}(b_j D_x{}^j v) \right] + \frac{v^2}{u} D_t u$$

$$= \frac{v}{u}(v\ell u - u m v) + \sum_{j=0}^{n} (b_j - a_j)(D_x{}^j v)^2$$

$$+ a_n \left(D_x{}^n v - \frac{D_x^{n-1} v}{D_x^{n-1} u} D_x{}^n u \right)^2$$

$$+ \sum_{k=1}^{n-1} \left(\frac{D_x{}^k v}{D_x{}^k u} - \frac{D_x^{k-1} v}{D_x^{k-1} u} \right)^2 (-1)^k D_x{}^k u \sum_{j=k}^{n} (-1)^j D_x^{j-k}(a_j D_x{}^j u). \tag{3}$$

Integrating both sides of (3) over $\{(x, t): x \in [x_1, x_2], t \in [0, \tau]\}$ and using (i)–(ii) of (1) and (i)–(ii) of (2), we have

$$\int_0^\tau \int_{x_1}^{x_2} (v^2/u)\, D_t u\, dx\, dt \geq \int_0^\tau \int_{x_1}^{x_2} \sum_{j=0}^n (b_j - a_j)\,(D_x^j v)^2\, dx\, dt,$$

which can be written as

$$\xi(\tau) - \xi(0) \geq \int_0^\tau \int_{x_1}^{x_2} \sum_{j=0}^n (b_j - a_j)\,(D_x^j v)^2\, dx\, dt,$$

where $\xi(t) = \int_{x_1}^{x_2} v^2 \ln |u|\, dx$. Consequently, by (iii) of (2), $\xi(t)$ tends to infinity as $t \to \infty$. It follows that u cannot be bounded in Q. This completes the proof.

By taking the operator $\mathscr{M}$ to be identically zero and arguing as in the proof of Theorem 1, we have the following variant of Theorem 1.

THEOREM 2. *Let u be as in Theorem 1. If there exists a function $v \in C^n[x_1, x_2]$ such that*

$$\begin{aligned} &v = D_x v = \cdots = D_x^{n-1} v = 0 \quad \textit{at} \quad x_1 \ \textit{and} \ x_2, \\ &\int_0^\tau \int_{x_1}^{x_2} \sum_{j=0}^n a_j (D_x^j v)^2\, dx\, dt \to -\infty \quad \textit{as} \quad \tau \to \infty, \end{aligned} \tag{4}$$

then u is unbounded in Q.

EXAMPLES. Let us consider the parabolic operator

$$\ell_1 u \equiv (-1)^n D_x^{2n} u - a_0 u + D_t u$$

in $Q_1 = \{(x, t): x \in (-1, 1), t > 0\}$, where a_0 is a positive constant. For the operator ℓ_1 we compute the integral in (4) with $v = (1 - x^2)^n$ to obtain

$$\int_0^\tau \int_{-1}^1 [(D_x^n v)^2 - a_0 v^2]\, dx\, dt = 2\tau[((2n)!!/(2n+1)) - ((4n)!!/(4n+1)!!)\, a_0],$$

so that the condition (4) is satisfied if a_0 is taken sufficiently large. In this case it follows from Theorem 2 that every function $u \in \mathfrak{D}_{\ell_1}$ satisfying (1) in Q_1 must be unbounded there.

In this connection, the necessity of the condition (ii) of (1) is shown by the function $u(x, t) = \exp(a_0^{1/2n}x)$, which is a bounded solution of $\ell_1 u = 0$ in Q_1 when n is even.

As another example, we consider the parabolic operator

$$\ell_2 u \equiv (-1)^n D_x^{2n} u + a_0 u + D_t u$$

in Q_1, where a_0 is again a positive constant. The integral in (4) associated with ℓ_2 is such that

$$\int_0^\tau \int_{-1}^1 [(D_x^n v)^2 + a_0 v^2] \, dx \, dt \to \infty \qquad \text{as} \qquad \tau \to \infty$$

for any nontrivial $v \in C^n[-1, 1]$ vanishing at -1 and 1 together with its first $n - 1$ derivatives. As is easily verified, the function

$$u(x, t) = \exp(-2a_0 t) \cdot \sin[a_0^{1/2n}(x - \alpha)]$$

is a solution of $\ell_2 u = 0$ satisfying the conditions (ii)–(iii) of (1) in the strip $\{(x, t); x \in (\alpha, \alpha + \pi/2a_0^{1/2n}), t > 0\}$, which contains Q_1 for suitably chosen a_0 and α. However, it is bounded in Q_1.

3. The Case of Matrix Differential Inequalities

In this section we wish to extend the results of Section 2 to the case of matrix differential inequalities. Consider the matrix differential operators

$$\mathscr{L} U \equiv \sum_{j=0}^{n} (-1)^j D_x^j(A_j(x, t) D_x^j U) + D_t U,$$

$$\mathscr{M} w \equiv \sum_{j=0}^{n} (-1)^j D_x^j(B_j(x) D_x^j w),$$

where A_j, B_j are $N \times N$ real symmetric matrix functions such that $A_j \in C^j(\bar{Q})$, $B_j \in C^j[x_1, x_2]$, $j = 0, 1, \ldots, n$. It is assumed further that the leading coefficients A_n and B_n are positive definite. The domain $\mathfrak{D}_{\mathscr{L}}$ of $\mathscr{L}$ is defined to be the set of all $N \times N$ real matrix functions U that are continuous in $\bar{Q}$ together with their first $2n$ x-derivatives and first t-derivatives. The domain $\mathfrak{D}_{\mathscr{M}}$ of $\mathscr{M}$ is the set of all real $N \times 1$ column vector functions w of class $C^{2n}[x_1, x_2]$.

Motivated by Chan and Young [1, 2] and Kusano and Yoshida [5], we introduce the following definition: An $N \times N$ matrix function V is said to be *prepared with respect to the operator* $\mathscr{L}$ (or simply $\mathscr{L}$*-prepared*) if

$$(D_x^{k-1}V)^{\mathrm{T}} \sum_{j=k}^{n} (-1)^j D_x^{j-k}(A_j D_x{}^j V) = \left(\sum_{j=k}^{n} (-1)^j D_x^{j-k}(A_j D_x{}^j V)\right)^{\mathrm{T}} D_x^{k-1} V,$$

$$(D_x{}^k V)^{\mathrm{T}} \sum_{j=k}^{n} (-1)^j D_x^{j-k}(A_j D_x{}^j V) = \left(\sum_{j=k}^{n} (-1)^j D_x^{j-k}(A_j D_x{}^j V)\right)^{\mathrm{T}} D_x{}^k V,$$

for $k = 1,..., n$, where the superscript T denotes taking the transpose of a matrix under consideration.

We remark that in the simplest case when the operator $\mathscr{L}$ is generated by a scalar operator l considered in Section 2, that is, when $A_j = a_j E_N$, $j = 0, 1,..., n$, E_N being the $N \times N$ identity matrix, the matrices of the forms $V = vE_N$ and $V = vI_N$, I_N being the $N \times N$ matrix with all the entries equal to 1, are prepared with respect to L.

THEOREM 3. *Let $U \in \mathfrak{D}_{\mathscr{L}}$ satisfy the following conditions:*

(i) *U is symmetric and L-prepared in Q,*

(ii) $U^{\mathrm{T}}\mathscr{L}U$ *and* $(-1)^k (D_x{}^k U)^{\mathrm{T}} \sum_{j=k}^{n} (-1)^j D_x^{j-k}(A_j D_x{}^j U)$,

$k = 1,..., n-1$, *are positive semidefinite in Q,* (5)

(iii) $U, D_x U,..., D_x^{n-1} U$ *are nonsingular in $\bar{Q}$,*

(iv) $D_t(\ln U)$ *commutes with* $\ln U$ *in Q, where the principal value of* $\ln U$ *is taken.*

If there exists a nontrivial $w \in \mathfrak{D}_{\mathscr{M}}$ which satisfies

(i) $\int_{x_1}^{x_2} w^{\mathrm{T}}\mathscr{M}w \, dx \leq 0$,

(ii) $w = D_x w = \cdots = D_x^{n-1} w = 0$ *at* x_1 *and* x_2, (6)

(iii) $\int_0^{\tau}\int_{x_1}^{x_2} \sum_{j=0}^{n} (D_x{}^j w)^{\mathrm{T}} (A_j - B_j) D_x{}^j w \, dx \, dt \to -\infty$ *as* $\tau \to \infty$,

then at least one of the eigenvalues of U is unbounded in Q.

Proof. Since U is $\mathscr{L}$-prepared and none of $U, D_x U,..., D_x^{n-1} U$ are singular in $\bar{Q}$, applying a generalized Picone identity given in [5, Theorem 1B], we obtain

$$\begin{aligned}
D_x \sum_{k=0}^{n-1} (-1)^k (D_x{}^k w)^{\mathrm{T}} \Bigg[\sum_{j=k+1}^{n} (-1)^j D_x^{j-k-1}(A_j D_x{}^j U)(D_x{}^k U)^{-1} D_x{}^k w \\
- \sum_{j=k+1}^{n} (-1)^j D_x^{j-k-1}(B_j D_x{}^j w)\Bigg] + w^{\mathrm{T}}(D_t U)\, U^{-1} w \\
= w^{\mathrm{T}}[(\mathscr{L}U)\, U^{-1} w - \mathscr{M} w] + \sum_{j=0}^{n} (D_x{}^j w)^{\mathrm{T}} (B_j - A_j) D_x{}^j w \qquad (7) \\
+ [D_x{}^n w - D_x{}^n U (D_x^{n-1} U)^{-1} D_x^{n-1} w]^{\mathrm{T}} \\
\times A_n [D_x{}^n w - D_x{}^n U (D_x^{n-1} U)^{-1} D_x^{n-1} w] \\
+ \sum_{k=1}^{n-1} [(D_x{}^k U)^{-1} D_x{}^k w - (D_x^{k-1} U)^{-1} D_x^{k-1} w]^{\mathrm{T}} \\
\times \left[(-1)^k (D_x{}^k U)^{\mathrm{T}} \sum_{j=k}^{n} (-1)^j D_x^{j-k}(A_j D_x{}^j U) \right] \\
\times [(D_x{}^k U)^{-1} D_x{}^k w - (D_x^{k-1} U)^{-1} D_x^{k-1} w].
\end{aligned}$$

Integrating (7) over $\{(x, t) : x \in [x_1, x_2],\ t \in [0, \tau]\}$ and using (ii) of (5) and (i)–(ii) of (6), we get

$$\int_0^\tau \int_{x_1}^{x_2} w^{\mathrm{T}}(D_t U)\, U^{-1} w \, dx\, dt \geqslant \int_0^\tau \int_{x_1}^{x_2} \sum_{j=0}^{n} (D_x{}^j w)^{\mathrm{T}} (B_j - A_j) D_x{}^j w \, dx\, dt. \qquad (8)$$

Defining $\eta(t) = \int_{x_1}^{x_2} w^T (\ln U)\, w \, dx$ and observing that $D_t(\ln U) = (D_t U)\, U^{-1}$ by (iv) of (5), we find from (8) that

$$\eta(\tau) - \eta(0) \geqslant \int_0^\tau \int_{x_1}^{x_2} \sum_{j=0}^{n} (D_x{}^j w)^{\mathrm{T}} (B_j - A_j) D_x{}^j w \, dx\, dt,$$

which together with (iii) of (6) implies that Re $\eta(t) \to \infty$ as $t \to \infty$.

Since U is real and symmetric, there exists an orthogonal matrix S such that

$$S U S^{\mathrm{T}} = \operatorname{diag}[\lambda_1, \ldots, \lambda_N],$$
$$S(\ln U)\, S^{\mathrm{T}} = \operatorname{diag}[\ln \lambda_1, \ldots, \ln \lambda_N],$$

where $\lambda_i = \lambda_i(x, t)$, $i = 1, \ldots, N$, are the eigenvalues of U; see, e.g., [3, pp. 65–66]. In terms of this S we obtain

$$\operatorname{Re} \eta(t) = \operatorname{Re} \int_{x_1}^{x_2} (Sw)^{\mathrm{T}} \, [S(\ln U)\, S^{\mathrm{T}}] \, (Sw) \, dx$$

$$= \sum_{i=1}^{N} \int_{x_1}^{x_2} \ln |\lambda_i(x, t)| \cdot [\zeta_i(x, t)]^2 \, dx,$$

where $\zeta_i(x, t)$ denotes the ith component of the column vector Sw. Consequently, at least one of $\lambda_i(x, t)$ must be unbounded in Q. The proof is therefore complete.

By taking the operator $\mathscr{M} \equiv 0$ we have the following variant of Theorem 3.

THEOREM 4. *Let U be as in Theorem 3. If there exists a vector function $w \in C^n[x_1, x_2]$ such that*

$$w = D_x w = \cdots = D_x^{n-1} w = 0 \qquad at \qquad x_1 \quad and \quad x_2, \tag{9}$$

$$\int_0^{\tau} \int_{x_1}^{x_2} \sum_{j=0}^{n} (D_x^j w)^{\mathrm{T}} A_j D_x^j w \, dx \, dt \to -\infty \qquad as \qquad \tau \to \infty,$$

then at least one of the eigenvalues of U must be unbounded in Q.

EXAMPLES. We consider the differential operator

$$\mathscr{L}_1 U \equiv (-1)^n D_x^{2n} U - a_0 U + D_t U$$

in $Q_1 = \{(x, t): x \in (-1, 1), t > 0\}$, where a_0 is a positive constant. If we take $w = ((1 - x^2)^n, \ldots, (1 - x^2)^n)^T$, then the integral in (9) becomes

$$\int_0^{\tau} \int_{-1}^{1} [| D_x^n w |^2 - a_0 | w |^2] \, dx \, dt$$

$$= 2N\tau[((2n)!!/(2n+1)) - ((4n)!!/(4n+1)!!)\, a_0].$$

Therefore, if a_0 is sufficiently large, it follows from Theorem 4 that every matrix function $U \in \mathfrak{D}_{\mathscr{L}_1}$ satisfying (5) has an eigenvalue which is unbounded in Q_1.

Consider the matrix function $U(x, t) = \exp(a_0^{1/2n} x)\, P$, where P is an $N \times N$ constant matrix which is symmetric and positive definite. It is obvious that $U(x, t)$ is a solution of $\mathscr{L}_1 U = 0$ in Q_1 when n is even and that all the eigenvalues of $U(x, t)$ are bounded in Q_1. This shows that the condition (ii) of (5) is necessary.

Finally, let us consider the differential operator

$$\mathscr{L}_2 U \equiv (-1)^n D_x^{2n} U + a_0 U + D_t U$$

in Q_1, where a_0 is a positive constant. The condition (9) is violated by any vector $w \in C^n[-1, 1]$ such that $w = D_x w = \cdots = D_x^{n-1} w = 0$ at -1 and 1. The matrix function $U(x, t)$ defined by

$$U(x, t) = \exp(-2a_0 t) \cdot \sin[a_0^{1/2n}(x - \alpha)]\, P,$$

P being as above, is a solution of $\mathscr{L}_2 U = 0$ in Q_1 provided a_0 and α are chosen appropriately. Although $U(x, t)$ satisfies (5) in Q_1, all the eigenvalues of $U(x, t)$ are bounded there.

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